Recitation 2 - Macroeconometrics

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March 3, 2018

Unit root and integration order

Definition

A process is said to be integrated of order d if it becomes stationary after being differenced d times.

Proposition

An AR(p) process with k unit roots z (or eigenvalues $\lambda = 1$) is integrated of order k.
(Reminder: $\lambda's$ are the reciprocal of the values z that solve the characteristic polynomial of the AR(p))

Example

Take the random walk: $X_t = X_{t-1} + \epsilon_t$ Its characteristic polynomial is $(1 - L)$, and the root is z $= 1$ The eigenvalue of the trivial AR(1) is $\lambda = 1$

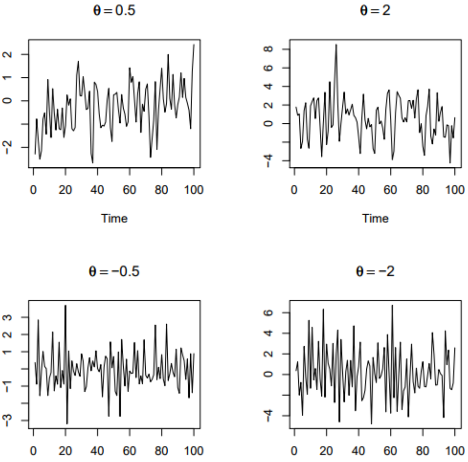
MA(1): $x_t = c + \epsilon_t + \theta\epsilon_{t-1}$

Condition for MA process: ϵ_t satisfies the conditions for WN: i) $E[\epsilon_t] = 0$, ii) $E[\epsilon_t^2] = \sigma^2$ constant, and iii) $E[\epsilon_t \epsilon_j] = 0$ for all j$\neq$t. Then we have:

- $E[x_t] = c$
- $Var[x_t] = (1 + \theta^2)\sigma^2$
- $Cov[x_t, x_{t-j}] = \theta\sigma^2$ if j=1, 0 if $j > 1$
- $Corr[x_t, x_{t-j}] = \frac{\theta}{(1+\theta^2)}$ if $j = 1$, 0 if $j > 1$

MA(1) is stationary for any θ. Why?

Stationarity of MA(1)



MA(q): $x_t = c + \epsilon_t + \theta_1 \epsilon_{t-1} + ... + \theta_q \epsilon_{t-q}$

An MA(q) is stationary for every sequence $\theta_1, \theta_2, ..., \theta_q$
$\rightarrow$ As proof, let's obtain its moments analytically:

- $E[x_t] = c$
- $Var[x_t] = (1 + \theta_1^2 + \theta_2^2 = .. + \theta_q^2)\sigma^2$
- $Cov[x_t, x_{t-j}] = (\theta_j + \theta_{j+1}\theta_1 + \theta_{j+2}\theta_2 + ... + \theta_q \theta_{q-j})\sigma^2$ if $j \leq q$, 0 if $j > q$

MA(∞): $x_t = c + \epsilon_t + \theta_1\epsilon_{t-1} + ... + \theta_\infty\epsilon_{t-\infty}$

- $x_t = c + \epsilon_t + \theta_1\epsilon_{t-1} + ... + \theta_\infty\epsilon_{t-\infty} = = \sum_{j=0}^{\infty} \theta_j \epsilon_{t-j}$

Definition

A sequence is **absolute summable** if $\sum_{i=0}^{\infty} |\alpha_i| < \infty$

Proposition

The $MA(\infty)$ is stationary if the coefficients are absolute summable.

Back to AR(p)

Proposition

If the characteristic polynomial of a AR(p) has roots $\lambda = 1$, it is not stationary.

Why? Recall that the AR(p) can be expressed as:
$(1 - \alpha L - \alpha_2 L^2 - ... - \alpha_p L^p)x_t =$
$(1 - \lambda_1^{-1} L)(1 - \lambda_2^{-1} L)...(1 - \lambda_p^{-1} L) = \epsilon_t$

And it has an $MA(\infty)$ representation if $\lambda_i^{-1} \neq 1$ (or $z_i \neq 1$) for all i: $x_t = \frac{1}{(1-\lambda_1^{-1} L)(1-\lambda_2^{-1} L)...(1-\lambda_p^{-1} L)} \epsilon_t$
Switching between AR and MA representations is called **inverting**.

ARMA (p,q)

ARMA(p,q): $\alpha_1 x_{t-1} + \alpha_2 x_{t-2} + ... + \alpha_p x_{t-p} + \theta_1 \epsilon_{t-1} + ... + \theta_q \epsilon_{t-q}$

It can be rewritten as: $A(L)x_t = \Theta(L)\epsilon_t$

Theorem

The ARMA(p,q) model is stationary provided the roots of the (L) polynomial lie outside the unit circle.

ARMA (p,q)

Intuition

→ AR part: For a series x_t, we can model that the level of its current observations depends on the level of its lagged observations. *Example: GDP*
→ MA part: we can model that the observations of a random variable at time t are not only affected by the shock at time t, but also the shocks that have taken place before. *Example: Economic impacts of an earthquake.*

Usually we can find multiple ways to represent a time series (through inverting parts of it). Which representation to choose depends on our problem:

- To study the impulse-response functions, MA representations are more convenient
- To estimate an ARMA model, AR representations maybe more convenient as usually x_t is observable while ϵ_t is not.

Invertibility

- By definition, $(1-\alpha L)^{-1}(1-\alpha L)=1$
- For the AR(1) process:
  - If we premulitply $(1-\alpha L)^{-1}$ to both sides of the equation, we get $x_t=(1-\alpha L)^{-1}\epsilon_t$.
  - Is there any explicit way to rewrite $(1-\alpha L)^{-1}$? We know that the answer turns out to be A(L).
  - Remember the transformation of AR(1) to $MA(\infty)$: It is done by defining $\theta_j=\alpha^j$.
- Similarly, for the AR(p) process, we can invert it if $|z_i|<1$ all the roots of A(z). We can permultiply by defining:
  $(1-z_1L)=A_1(L)$
  $(1-z_2L)=A_2(L)$, etc...
- For MA(1) to be invertible we need that $|\theta|<1$

ARIMA (p,d,q)

- This is the most general family of models. A stationary ARMA(1,1) model is an ARIMA(1,0,1).
- The parameter d represents the order of integration. This is, the number of times the series has to be differenced to become stationary.
  If d=0: $y_t = Y_t$
  If d=1: $y_t = Y_t - Y_{t-1}$
  If d=2:
  $y_t = (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}) = Y_t - 2Y_{t-1} + Y_{t-2}$

Definition: ARIMA (p,d,q)

ARIMA(p,d,q): $A(L)\Delta^d x_t = \Theta(L)\epsilon_t$

ARIMA (p,d,q)

Example (Special cases)

- White noise: ARIMA(0,0,0) $X_t = \varepsilon_t$
- Random walk : ARIMA(0,1,0): $\Delta X_t = \varepsilon_t \Rightarrow X_t = X_{t-1} + \varepsilon_t$