Recitation 1 - Macroeconometrics

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February 14, 2018

Time Series

$x_1, x_2, ..., x_T$ or $x_t, t = 1, ..., T$
Where x_t is a **random variable**. A sequence of random variables is a **stochastic process**, defined by a joint distribution function. In its most general form:

$$x_t = f(x_{t-1}, x_{t-2}, ..., u_t)$$

Examples of f: $x_t = \epsilon_t$ or $x_t = \alpha x_{t-1} + \epsilon_t$, where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$

- We can separate time series into two categories:
  - Univariate ($x_t \in \mathbb{R}$ is scalar):The primary model for univariate time series is autoregressions (ARs).
  - Multivariate ($x_t \in \mathbb{R}^m$ is vector-valued): The primary model for multivariate time series is vector autoregressions (VARs).

White noise

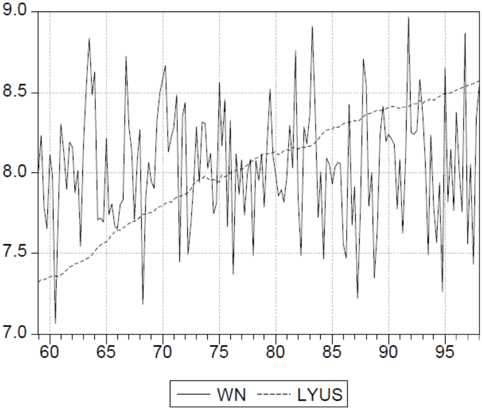

FIG. 2.1. A *white-noise* process and the log of US real disposable income

→ White noise model (WN), does not capture **persistence**, therefore it is not a proper model for most macroeconomics time-series.

Types of Univariate Autoregressions (AR)

We are in a different world from microeconometrics. Classical assumptions are not valid:

- Because of the sequential nature of time series, we expect that x_t and x_{t-1} are not independent
- Instead of that we are going to model different ways in which x_t and x_{t-1} can be dependent

We will first concentrate in a class of models created by taking linear combinations of white noise:

- AR(1) :$x_t = \rho x_{t-1} + \epsilon_t$
- MA(1) :$x_t = \epsilon_t + \theta \epsilon_{t-1}$
- AR(p) :$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + ... + \rho_p x_{t-p} + \epsilon_t$
- MA(q) :$x_t = \epsilon_t + \theta_1 \epsilon_{t-1} + ... + \theta_q \epsilon_{t-q}$
- ARMA(p,q): $\rho_1 x_{t-1} + \rho_2 x_{t-2} + ... + \rho_p x_{t-p} + \theta_1 \epsilon_{t-1} + ... + \theta_q \epsilon_{t-q}$

Conditional and unconditional densities

The density function can be characterized by its moments.
Conditional moments:

- $E[x_t | x_t - 1]$
- $Var[x_t | x_t - 1]$
- $Cov[(x_t | x_t - 1), (x_{t-j} | x_{t-j-1})]$

Unconditional moments:

- $E[x_t]$
- $Var[x_t]$
- $\gamma(j) = Cov(x_t, x_{t-j})$
- $\rho(j) = \frac{Cov(x_t, x_{t-j})}{\sqrt{Var[x_t] Var[x_{t-1}]}}$

Example: AR model

Obtain the conditional and unconditional moments of the following AR(1) model.

- $x_t = \alpha_1 x_{t-1} + \epsilon_t$, where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$

Stationarity

Covariance stationarity:

- x_t is **covariance (weakly) stationary** if $E(x_t) = \mu$ is independent of t and $cov(x_t, x_{t-j}) = \gamma(j)$ is independent of t -i.e. depends only on the lag j-.
- $\gamma(j)$ is called the **autocovariance function**.
- $\rho(j) = \gamma(j)/\gamma(0) = corr(x_t, x_{t-j})$ is the **autocorrelation function**.

Strict stationarity:

- x_t is **strictly stationary** if the joint distribution of (x_t, x_{t-j}) is independent of t for all j.

A stationary time series is ergodic if $\gamma(j) \to 0$ as $j \to \infty$.

AR(1) Process

A mean-zero AR(1) is $x_t = \alpha_1 x_{t-1} + \epsilon_t$
Assume that ϵ_t is iid, $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) = \sigma^2$

By back-substitution, we find
$x_t = \epsilon_t + \alpha\epsilon_{t-1} + \alpha^2\epsilon_{t-2} + ...$
$\Rightarrow x_t = \sum_{j=0}^{\infty} \alpha^j \epsilon_{t-j}$
Loosely speaking, this series converges if the sequence $\alpha^j \epsilon_{t-j}$ gets small as $j \to \infty$ This occurs when $|\alpha| < 1$

Theorem 1. If and only if $|\alpha| < 1$ then x_t is strictly stationary and ergodic.

Important points from AR(1):
- It can be expressed as a MA(∞)
- If stationary, the unconditional mean can be calculated without knowing x_0

The lag operator

The lag operator L satisfies $Lx_t = x_{t-1}$. It is easy to see:

- $L^2 x_t = L x_{t-1}$
- Rewriting, AR(1) model: $x_t - \alpha x_{t-1} = \epsilon_t \rightarrow x_t(1 - \alpha L) = \epsilon_t$

We call $\alpha(L) = (1 - \alpha L)$ the **autoregressive polynomial of** x_t .
We say that the **root of the polynomial** is $1/\alpha$. since $p(z) = 0$ when $z = 1/\alpha$.
AR(1) is stationary *iff* the root of the AR polynomial is larger than one in absolute value $|z| > 1$ (equivalent to Theorem 1 !)

Stationarity of AR(p)

AR(p) :$x_t = \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + ... + \alpha_p x_{t-p} + \epsilon_t$

Using the lag operator:
$x_t = \alpha_1 L x_t + \alpha_2 L^2 x_t + ... + \alpha_p L^p x_t + \epsilon_t$
$\Rightarrow \alpha(L) y_t = \epsilon_t$

We define: $p(L) = 1 - \alpha L - \alpha_2 L^2 - ... - \alpha_t L^p$

The **Fundamental Theorem of Algebra** applied to the lag polynomial implies that it can always be factored as:
$p(L) = (1 - \lambda_1^{-1} L)(1 - \lambda_2^{-1} L)...(1 - \lambda_p^{-1} L)$

Where the $\lambda_1, ..., \lambda_k$ a are the complex roots of $p(L)$ which satisfy $p(L) = 0$.
$\rightarrow$ We know that an AR(1) is stationary iff the absolute value of the root of its autoregressive polynomial is larger than one. For an AR(k), the requirement is that all roots are larger than one.